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A STUDY OF THE STABILITY OF REINFORCED CYLINDRICAL AND CONICAL SHELLS SUBJECTED TO VARIOUS TYPES AND COMBINATIONS OF LOADS

SECTION I - General Instability of an Orthotropic Circular Cylindrical Shell Subjected to a Pressure Combined with an Axial Load Considering Both Clamped and Simply Supported Edge Conditions

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GENERAL INSTABILITY OF AN ORTHOTROPIC CIRCULAR CYLINDRICAL SHELL SUBJECTED TO A PRESSURE COMBINED WITH AN AXIAL LOAD CONSIDERING BOTH CLAMPED AND SIMPLY SUPPORTED EDGE CONDITIONS

By
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INTRODUCTION

In a discussion of the instability of a circular cylindrical shell the term "short" is generally applied to a cylinder whose length is approximately equal to the radius. When a short cylindrical shell is subjected to an axial load that produces instability, the effect of the boundary conditions is no longer insignificant. In the analyses presently available, such as References (1) and (2), the solution to the Donnell type of differential equation for an orthotropic circular cylindrical shell is found for simply supported edge conditions. Therefore, a definite need exists for the development of an expression that will yield the buckling criteria in a form usable for designers when a short orthotropic or stiffened cylindrical shell is subjected to a combination of a pressure and an axial load and has boundary conditions other than simply supported.

In addition, for very thin circular cylindrical shell, with a radius to thickness ratio greater than 200, the so-called small deflection or linear theory does not yield satisfactory agreement with experimental results.

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Non-linear analyses based on an assumed deflection pattern, such as References (3), (4), and (5), indicate better agreement with test data; but such analyses as now constituted are not applicable for design criteria. In this paper a constant factor is included in a radial displacement expression for the purpose of linearizing certain energy terms that were neglected in Reference (2) so that their effect may be studied. Such an approach or similar one is needed in instability shell studies for as presently constituted the the inclusion of non-linear terms in such studies results in such complex mathematical procedures that their practical application is almost prohibitive. The results of such attempted linearization as mentioned above will be presented in another paper.

In order to develop the most complete analysis possible based on existing strain-displacement information, the investigation was begun by using the best general theoretical analysis and modifying and limiting it when needed as indicated by the mathematical difficulties encountered. The theoretical approach used is similar to the one used in Reference (2) wherein orthotropic shell analysis is applied to a stiffened circular cylinder; however, additional strain-displacement terms are included that will later make possible the linearization study mentioned above.

In the present paper a set of instability equilibrium equations, similar to those of Reference (2), are derived for an orthotropic circular cylindrical shell by applying variational methods to the expression for the total energy of the shell. From these equilibrium equations an eighth order differential equation of the Donnell type is obtained for a cylinder of uniform thickness subjected to a pressure and a compressive axial force. This differential equation is solved for the case of simply supported edge conditions, and a quadratic algebraic expression is developed that yields the buckling criteria. This algebraic expression can be minimized quite readily for certain parameters by the use of a digital computer for the purpose of establishing design criteria. The Donnell type differential equation is also solved for the case of clamped edge conditions, and a four-by-four determinant that yields the buckling criteria is developed. Again it may be possible that design criteria may be established by minimizing this determinant for certain parameters by using a digital computer.

STRESS-STRAIN RELATIONS

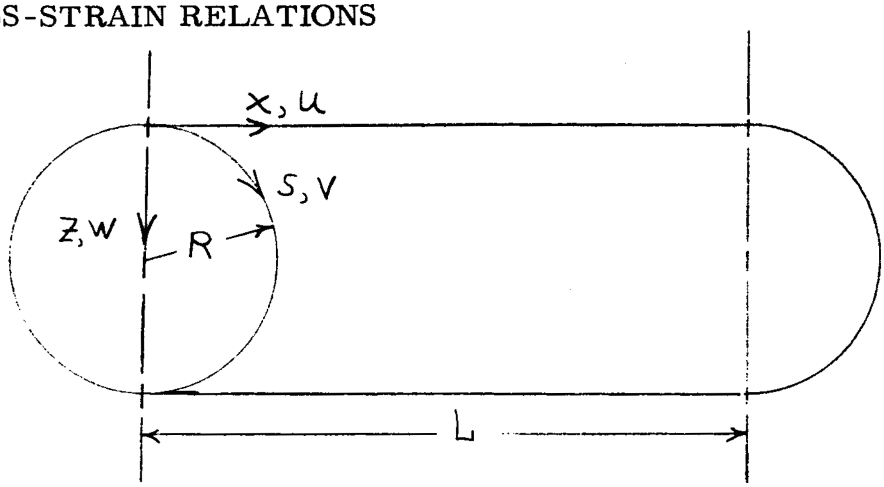

Figure 1: Coordinate System and Displacements of Circular Cylindrical Shell

The circular cylindrical shell geometry employed, which is the same as that of Reference (2), is shown in Figure (1) together with the coordinate system used and the corresponding middle-surface displacements. In terms of the shell middle-surface displacements, u , v , and w , the expressions for the buckling strains in the shell wall are the same as those given in References (2) with some additional terms. The strain-displacement relationships used are written as follows.

$$
\begin{aligned}
e_{xx} &= u_{,x} + \tfrac{1}{2} w_{,x}^{2} - z\, w_{,xx} \\
e_{ss} &= v_{,s} - w/R + \tfrac{1}{2}\left(w_{,s} + v/R\right) - z\left(w_{,ss} + K\, w/R^{2}\right) \qquad (1) \\
e_{xs} &= \tfrac{1}{2}\left[v_{,x} + u_{,s} + \left(w_{,s} + v/R\right) w_{,x} - (z/2)\left(2 w_{,xs} - v_{,x}/R - u_{,s}/R\right)\right]
\end{aligned}
$$

where K is a constant, R is the radius of the cylinder; e_{xx} , e_{ss} , and e_{xs} , are the axial, circumferential, and shear strains, respectively; and a comma indicates differentiation with respect to the succeeding variable.

For a homogeneous orthotropic material, the stress-strain relations in generalized plane stress can be written as follows.

$$\sigma_{xx} = E_x(\epsilon_{xx} - \nu_{sx}\epsilon_{ss})/(1 - \nu_{sx}\nu_{xs})$$

$$\sigma_{ss} = E_s(\epsilon_{ss} - \nu_{xs}\epsilon_{xx})/(1 - \nu_{sx}\nu_{xs}) \tag{2}$$

$$\sigma_{xs} = G\,\epsilon_{xs}$$

In the preceding equations σ_{xx}, σ_{ss}, and σ_{xs}, are the axial, circumferential, and shear stresses, respectively; E_x and E_s are the values of the moduli of elasticity averaged over the thickness in the axial and circumferential directions, respectively; G is the average shear modulus, and ν_{sx} and ν_{xs} are Poisson's ratios.

For convenience in later calculations, the following constants and notations, similar to those given in Reference (2), are introduced.

$$\alpha_1 = (E_x h)/(1 - \nu_{xs}\nu_{sx}) \qquad D_1 = (E_x h^3/12)/(1 - \nu_{sx}\nu_{xs})$$

$$\alpha_2 = (E_s h)/(1 - \nu_{xs}\nu_{sx}) \qquad D_2 = (E_s h^3/12)/(1 - \nu_{xs}\nu_{sx})$$

$$\alpha_3 = G h \qquad D_3 = G h^3/12 \tag{3}$$

$$\alpha_4 = (E_x \nu_{sx} h)/(1 - \nu_{sx}\nu_{xs}) \qquad D_4 = (E_x \nu_{sx} h^3/12)/(1 - \nu_{sx}\nu_{xs})$$

$$= (E_s \nu_{xs} h)/(1 - \nu_{xs}\nu_{sx}) \qquad = (E_s \nu_{xs} h^3/12)/(1 - \nu_{sx}\nu_{xs})$$

where h is the thickness of the shell. The following stress resultants are also defined.

$$\bar{N}_{xx} = \int_{-h/2}^{h/2} \bar{\sigma}_{xx}\, dz \qquad \bar{N}_{ss} = \int_{-h/2}^{h/2} \bar{\sigma}_{ss}\, dz \qquad \bar{N}_{xs} = \int_{-h/2}^{h/2} \bar{\sigma}_{xs}\, dz \tag{4}$$

Based on Maxwell's reciprocal theorem, the following relationship must hold between the elastic constants.

$$E_s \nu_{xs} = E_x \nu_{sx} \tag{5}$$

The two expressions for α_4 and D_4 in Equations (3) are the result of the above relationship.

STRAIN ENERGY AND TOTAL ENERGY EXPRESSION

The instability differential equations of equilibrium will be derived using the same procedure as given in Reference (2). For an elastic system, a criterion of buckling is that the variation of the change in the energy of the system due to buckling, with respect to the displacements, must be zero. Described mathematically, this criterion becomes

$$\delta(\mathcal{U}+\mathcal{V})=0 \tag{6}$$

where $\mathcal{U}$ is the change in the strain energy of the shell and $\mathcal{V}$ is the change in the potential energy of the external forces during the buckling process.

If initial bending stresses are neglected

$$\mathcal{U}=\int_{V_s}\left[\left(\bar{\sigma}_{xx}e_{xx}+\bar{\sigma}_{ss}e_{ss}+\bar{\sigma}_{xs}e_{xs}\right)+\frac{1}{2}\left(\sigma_{xx}e_{xx}+\sigma_{ss}e_{ss}+\sigma_{xs}e_{xs}\right)\right]dV_s \tag{7}$$

In the preceding expression $\bar{\sigma}_{xx}$, $\bar{\sigma}_{ss}$, and $\bar{\sigma}_{xs}$ are the membrane stresses existing in the shell in the compressed but unbuckled state; σ_{xx} , σ_{ss} , and σ_{xs} are the stresses superimposed during the buckling process, and V_s is the volume of the shell wall.

The strain energy of the shell can be computed in terms of the buckling strains and the pre-buckling stresses by substituting Equation (2) into Equation (8) with the following result.

$$\begin{aligned}\mathcal{U}=\frac{1}{2}\int_{V_s}&\left[2\left(\bar{\sigma}_{xx}e_{xx}+\bar{\sigma}_{ss}e_{ss}+2\bar{\sigma}_{xs}e_{xs}\right)+E_x e_{xx}^2/(1-\nu_{xs}\nu_{sx})\right.\\&\left.+E_s e_{ss}^2/(1-\nu_{xs}\nu_{sx})+2E_x\nu_{sx}e_{xx}e_{ss}/(1-\nu_{xs}\nu_{sx})+2G e_{xs}^2\right]dV_s\end{aligned} \tag{8}$$

If p designates a radial pressure, then $\bar{N}_{ss}=-pR$ for the case of an external pressure; and $\bar{N}_{ss}=pR$ for an internal pressure. The theoretical development will be continued for $\bar{N}_{ss}=-pR$, and for this case the change in the potential energy of the external forces during buckling is given by the following expression.

$$\mathcal{V}=-\int_{A_s}\left[pR\left(w/R-v_{,s}\right)+\bar{N}_{xx}u_{,x}+\bar{N}_{xs}\left(v_{,x}+u_{,s}\right)\right]dA_s \tag{9}$$

where A_s is the middle surface area of the shell wall.

The total energy of an orthotropic circular cylindrical shell can then be obtained in terms of the displacements and their derivatives by substituting Equations (1) into Equation (8) and adding the result to Equation (9). After integrating over the shell thickness and retaining only second order terms, the following expression for the total energy is obtained.

$$
\begin{aligned}
U + V = \int_{A_s} \{ & \bar{N}_{xx}[u_{,x} + (\tfrac{1}{2}) w_{,x}^2] + \bar{N}_{xs}[u_{,s} + v_{,x} + w_{,x} w_{,s} + v w_{,x}/R] \\
& - pR[v_{,s} - w/R + (\tfrac{1}{2})(w_{,s}^2 + 2v w_{,s}/R + v^2/R^2)]\} \, dA_s \\
& + (h/2)\int_{A_s} \{[1/(1-\nu_{xs}\nu_{sx})][E_x u_{,x}^2 + E_s v_{,s}^2 - 2E_s v_{,s} w/R + E_s w^2/R \\
& + 2 E_x \nu_{sx}(v_{,s} u_{,x} - u_{,x} w/R)] + (G/2)[v_{,x}^2 + u_{,s}^2 + 2v_{,x} u_{,s}]\} dA_s \\
& + (h^3/24)\int_{A_s} \{[1/(1-\nu_{xs}\nu_{sx})][E_x w_{,xx}^2 + E_s w_{,ss}^2 + 2E_s K w w_{,ss}/R^2 \\
& + K^2 w^2/R^4 + 2E_x \nu_{sx} w_{,xx} w_{,ss} + 2E_x \nu_{sx} K w_{,xx} w/R^2] \\
& + (G/2)[4 w_{,xs}^2 + v_{,x}^2/R^2 + u_{,s}^2/R^2 + 4w_{,xs} v_{,s}/R - 4 w_{,xs} u_{,s}/R - 2v_{,x} u_{,s}/R]\} dA_s \\
& - \int_{A_s} [pR(w/R - v_{,s}) + \bar{N}_{xx} u_{,x} + \bar{N}_{xs}(v_{,x} + u_{,s})] \, dA_s
\end{aligned} \tag{10}
$$

EQUILIBRIUM EQUATIONS AND NATURAL BOUNDARY CONDITIONS RESULTING FROM THE APPLICATION OF VARIATIONAL PROCEDURES

From Equation (10) the following expression is obtained for the variation in the total energy of the shell after making the substitutions indicated by Equations (3).

$$
\begin{aligned}
\delta(U + V) = \int_{A_s} \{ & \bar{N}_{xx}[w_{,x} \delta w_{,x}] - pR[(w_{,s} + v/R)\delta w_{,s} + (w_{,s}/R + v/R^2)\delta v] \\
& + \bar{N}_{xs}[(w_{,s} + v/R)\delta w_{,x} + w_{,x}\delta w_{,s} + (w_{,x}/R)\delta v]\} \, dA_s \\
& + \int_{A_s} \{[\alpha_1 u_{,x} + \alpha_4(v_{,s} - w/R)]\delta u_{,x} + [\alpha_3(u_{,s} + v_{,x})]\delta u_{,s} \\
& + [\alpha_3(u_{,s} + v_{,x})]\delta v_{,x} + [\alpha_2(v_{,s} - w/R) + (\alpha_4 u_{,x})]\delta v_{,s} \\
& + [\alpha_2(w/R^2 - v_{,s}/R) - (\alpha_4 u_{,x}/R)]\delta w\} \, dA_s \\
& + \int_{A_s} \{[D_1 w_{,xx} + D_4(w_{,ss} - Kw/R^2)]\delta w_{,xx} + D_3[2w_{,xs} + v_{,x}/R - u_{,s}/R]\delta w_{,xs} \\
& + [D_2(w_{,ss} + Kw/R^2) + D_4 w_{,xx}]\delta w_{,ss} + [D_2(K w_{,ss}/R^2 + K^2 w/R^4) + D_4 K w_{,xx}/R^2]\delta w \\
& + [D_3(u_{,s}/2R^2 - w_{,xs}/R - v_{,x}/2R^2)]\delta u_{,s} + [D_3(v_{,x}/2R^2 + w_{,xs}/R - u_{,s}/2R^2)]\delta v_{,x}\} \, dA_s
\end{aligned} \tag{11}
$$

After using $dA_s = dx\,ds$, applying a well-known identity from the calculus of variations, and integrating Equation (11) by parts between the proper limits, the following expression is obtained for the variation in the total energy.

$$
\begin{aligned}
\delta(U+V)=\int_0^L\!\!\int_0^{2\pi}\Big\{&\big[\alpha_4(w_{,x}/R-v_{,sx})-\alpha_1 u_{,xx}-\alpha_3(u_{,ss}+v_{,xs})+D_3(w_{,xss}/R+v_{,xs}/2R^2-u_{,ss}/2R^2)\big]\delta u\\
&+\big[\bar{N}_{xs}w_{,x}/R-p(w_{,s}+v/R)+\alpha_2(w_{,s}/R-v_{,ss})-\alpha_4 u_{,xs}-\alpha_3(v_{,xx}+u_{,sx})\\
&+D_3(u_{,sx}/2R^2-v_{,xx}/2R^2-w_{,xxs}/R)\big]\delta v\\
&+\big[p(Rw_{,ss}+v_{,s})-\bar{N}_{xx}w_{,xx}-\bar{N}_{xs}(2w_{,xs}+v_{,x}/R)+\alpha_2(w/R^2-v_{,s}/R)\\
&-\alpha_4 u_{,x}/R+D_1 w_{,xxxx}+D_2(w_{,ssss}+2Kw_{,ss}/R^2+K^4 w/R^4)\\
&+2D_4(w_{,xxss}+Kw_{,xx}/R^2)+D_3(2w_{,xxss}+v_{,xxs}/R-u_{,xxs}/R)\big]\delta w\Big\}\,dx\,ds \qquad (12)\\
+\int_0^{2\pi R}\Big\{&\big[\alpha_1 u_{,x}+\alpha_4(v_{,s}-w/R)\big]_0^L\delta u+\big[\alpha_3(v_{,x}+u_{,s})+D_3(v_{,x}/2R^2+w_{,xs}/R-u_{,s}/2R^2)\big]_0^L\delta v\\
&+\big[\bar{N}_{xx}w_{,x}+\bar{N}_{xs}(w_{,s}+v/R)-D_1 w_{,xxx}+D_3(u_{,ss}/R-v_{,xs}/R-2w_{,xss})\\
&-D_4(w_{,ssx}+Kw_{,x}/R^2)\big]_0^L\delta w+\big[D_1 w_{,xx}+D_4(w_{,ss}+Kw/R^2)\big]_0^L\delta w_{,x}\Big\}\,ds\\
+\int_0^L\Big\{&\big[\alpha_3(u_{,s}+v_{,x})+D_3(u_{,s}/2R^2-w_{,xs}/R-v_{,x}/2R^2)\big]_0^{2\pi R}+\big[\alpha_2(v_{,s}-w/R)+\alpha_4 u_{,x}\big]_0^{2\pi R}\delta v\\
&+\big[\bar{N}_{xs}w_{,x}-pR(w_{,s}+v/R)-D_2(w_{,sss}+Kw_{,ss}/R)+D_3(u_{,sx}/R-v_{,xx}/R-2w_{,xxs})\\
&-D_4 w_{,xxs}\big]_0^{2\pi R}\delta w+\big[D_2 w_{,ss}+Kw/R^2+D_4 w_{,xx}\big]_0^{2\pi R}\delta w_{,s}\Big\}\,dx\\
&+\Big[D_3(2w_{,xs}+v_{,x}/R-u_{,s}/R)\Big]_0^L\Big]_0^{2\pi R}\delta w
\end{aligned}
$$

The change in the total energy of a system must vanish for any of the arbitrary virtual displacements δu, δv, and δw when the system is in equilibrium. Therefore the integrands in the surface integral of Equation (12) that are multiplied by δu, δv, and δw, respectively, must vanish; and the following equations of equilibrium are obtained from this requirement.

$$
\alpha_1 u_{,xx}+(\alpha_3+D_3/2R^2)u_{,ss}+(\alpha_4+\alpha_3-D_3/2R^2)v_{,xs}-\alpha_4 w_{,x}/R-D_3 w_{,xss}/R=0 \qquad (13)
$$

$$
\begin{aligned}
pv/R-(p-\alpha_2/R)w_{,s}-\bar{N}_{xs}w_{,x}/R+(\alpha_3+D_3/2R^2)v_{,xx}+\alpha_2 v_{,ss}&\\
+(\alpha_4+\alpha_3-D_3/2R^2)u_{,sx}+D_3 w_{,xxs}/R=0& \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
(\alpha_2/R^2+K^2D_2/R^4)w+(2D_4K/R^2-\bar{N}_{xx})w_{,xx}+(pR+2KD_2/R^2)w_{,ss}&\\
-2\bar{N}_{xs}w_{,xs}+D_1 w_{,xxxx}+D_2 w_{,ssss}+(2D_3+2D_4)w_{,xxss}-\alpha_4 u_{,x}/R&\\
-\bar{N}_{xs}v_{,x}/R+(p-\alpha_2/R)v_{,s}+D_3 v_{,xxs}/R-D_3 u_{,xss}/R=0& \qquad (15)
\end{aligned}
$$

The following natural boundary conditions are obtained as a result of the requirement that the change in the total energy of the system represented by the line integrals of Equation (12) must vanish for any of the virtual displacements or their derivatives.

$$\alpha_1 u_{,x} + \alpha_4 (v_{,s} - w/R)\Big]_0^L = 0 \qquad \text{or} \quad \delta u\Big]_0^L = 0$$

$$\alpha_2 (u_{,s} + v_{,x}) + D_3 (u_{,s}/2R^2 + w_{,xs}/R - v_{,x}/2R^2)\Big]_0^{2\pi R} = 0 \qquad \text{or} \quad \delta u\Big]_0^{2\pi R} = 0$$

$$\alpha_3 (v_{,x} + u_{,s}) + D_3 (v_{,x}/2R^2 + w_{,xs}/R - u_{,s}/2R^2)\Big]_0^L = 0 \qquad \text{or} \quad \delta v\Big]_0^L = 0$$

$$\alpha_2 (v_{,s} - w/R) + \alpha_4 u_{,x}\Big]_0^{2\pi R} \qquad \text{or} \quad \delta v\Big]_0^L = 0$$

$$\bar{N}_{xx} w_{,x} + \bar{N}_{xs}(w_{,s} + v/R) - D_1 w_{,xxx} - D_4 (w_{,ssx} + K w_{,x}/R^2)$$
$$+ D_3 (u_{,ss}/R - v_{,xs}/R - 2 w_{,xss})\Big]_0^L = 0 \qquad \text{or} \quad \delta w\Big]_0^L = 0$$

$$\bar{N}_{xs} w_{,x} - pR(w_{,s} + v/R) - D_2 (w_{,sss} + K w_{,s}/R^2) - D_4 w_{,xxs}$$
$$+ D_3 (u_{,sx}/R - v_{,xx}/R - 2 w_{,xxs})\Big]_0^{2\pi R} = 0 \qquad \text{or} \quad \delta w\Big]_0^{2\pi R} = 0$$

$$D_3 (2 w_{,xs} + v_{,x}/R - u_{,s}/R)\Big]_0^L\Big]_0^{2\pi R} = 0 \qquad \text{or} \quad \delta w\Big]_0^L\Big]_0^{2\pi R} = 0$$

$$D_1 w_{,xx} + D_4 (w_{,ss} + K w/R^2)\Big]_0^L \qquad \text{or} \quad \delta w_{,x}\Big]_0^L = 0$$

$$D_2 (w_{,ss} + K w/R^2) + D_4 w_{,xx}\Big]_0^{2\pi R} \qquad \text{or} \quad \delta w_{,s}\Big]_0^{2\pi R} = 0$$

DEVELOPMENT OF A DONNELL TYPE DIFFERENTIAL EQUATION

In order to derive a Donnell type of differential equation, Equations (13), (14), and (15) are written in the following manner.

$$v_{,xs} = a_1 u_{,xx} + a_2 u_{,ss} + a_3 w_{,x} + a_5 w_{,xss} \tag{13a}$$

$$u_{,xs} = b_0 v + b_1 v_{,xx} + b_2 v_{,ss} + b_3 w_{,x} + b_4 w_{,s} + b_5 w_{,xxs} \tag{14a}$$

$$c_1 u_{,x} + c_2 v_{,x} + c_3 v_{,s} + c_4 u_{,xss} + c_5 v_{,xxs} - c_6 w - c_7 w_{,xx}$$
$$- c_8 w_{,ss} - c_9 w_{,xs} - D_1 w_{,xxxx} - D_2 w_{,ssss} - 2(D_3 + D_4) w_{,xxss} = 0 \tag{15a}$$

where

$$d = \alpha_2 + \alpha_1 - D_3/2R^2 \qquad a_1 = -\alpha_1/d \qquad a_2 = -(2\alpha_3 R^2 + D_3)/(2R^2 d)$$
$$a_3 = \alpha_4/Rd \qquad a_5 = D_3/Rd \qquad b_0 = -p/Rd$$
$$b_1 = -(2\alpha_3 R^2 + D_3)/2R^2 d \qquad b_2 = -\alpha_2/d \qquad b_3 = \bar{N}_{xs}/Rd$$
$$b_4 = (\alpha_2 - pR)/Rd \qquad b_5 = -D_3/Rd \qquad c_1 = \alpha_4/R \tag{16}$$

$$C_2 = \bar{N}_{xs}/R \qquad C_3 = (\alpha_2 - pR)/R \qquad C_4 = D_3/R$$

$$C_5 = -D_3/R \qquad C_6 = (\alpha_2 R^2 + K^2 D_2)/R^4 \qquad C_7 = (2KD_4 - R^2\bar{N}_{xx})/R^2$$

$$C_8 = (pR^3 + 2D_2K)/R^2 \qquad C_9 = -2\bar{N}_{xs}$$

A linear differential operator is defined as follows:

$$Q = a_1 b_0 \frac{\partial^2}{\partial x^2} + a_2 b_0 \frac{\partial^2}{\partial s^2} + a_1 b_1 \frac{\partial^4}{\partial x^4} + (a_1 b_2 + a_2 b_1 - 1)\frac{\partial^4}{\partial x^2 \partial s^2} + a_2 b_2 \frac{\partial^4}{\partial s^4} \tag{17}$$

By successive differentiation and combination Equations (13a) and (14a) can be brought into the following form.

$$-Qu = a_3 b_0 w_{,x} + a_3 b_1 w_{,xxx} + (a_5 b_0 + a_3 b_2 + b_4) w_{,xss} + b_3 w_{,xxs} + (a_5 b_1 + b_5) w_{,xxxss} + a_5 b_2 w_{,xssss} \tag{18}$$

$$-Qv = a_1 b_3 w_{,xxx} + (a_1 b_4 + a_3) w_{,xxs} + a_2 b_3 w_{,xss} + a_2 b_4 w_{,sss} + (a_2 b_5 + a_5) w_{,xxsss} + a_1 b_5 w_{,xxxxs} \tag{19}$$

Operating on Equation (15a) with Q results in the succeeding expression.

$$-Q(c_1 u_{,x} + c_2 v_{,x} + c_3 v_{,s} + c_4 u_{,xss} + c_5 v_{,xxs}) + Q[c_6 w + c_7 w_{,xx} + c_8 w_{,ss} + c_9 w_{,xs} + D_1 w_{,xxxx} + D_2 w_{,ssss} + 2(D_3 + D_4) w_{,xxss}] = 0 \tag{15b}$$

All the u and v terms in Equation (15b), by utilizing Equations (18) and (19), can be eliminated with the result that the following eight order Donnell-type differential equation in w alone is obtained.

$$(a_2 b_2 D_2) w_{,ssssssss} + (a_5 b_2 c_4 + a_1 b_2 D_2 + a_2 b_1 D_2 - D_2 + 2a_2 b_2 D_4 + 2a_2 b_2 D_3) w_{,xxssssss}$$

$$+ (a_5 b_1 c_4 + b_5 c_4 + a_2 b_5 c_5 + a_5 c_5 + a_2 b_2 D_1 + a_1 b_1 D_2 + 2a_1 b_2 D_3 + 2a_1 b_2 D_4$$

$$+ 2a_2 b_1 D_3 + 2a_2 b_1 D_4 - 2D_3 - 2D_4) w_{,xxxxssss} + (a_1 b_5 c_5 + a_1 b_2 D_1$$

$$+ a_2 b_1 D_1 - D_1 + 2a_1 b_1 D_3 + 2a_1 b_1 D_4) w_{,xxxxxxss} + (a_1 b_1 D_1) w_{,xxxxxxxx}$$

$$+ a_2 b_2 c_8 + a_2 b_0 D_2) w_{,ssssss} + (a_2 b_2 c_9) w_{,xsssss} + (a_5 b_2 c_1 + a_2 b_5 c_3$$

$$+ a_5 c_3 + b_4 c_4 + a_5 b_0 c_4 + a_3 b_2 c_4 + a_2 b_4 c_5 + a_2 b_2 c_7 + a_1 b_2 c_8 + a_2 b_1 c_8$$

$$- c_8 + a_1 b_0 D_2 + 2a_2 b_0 D_3 + 2a_2 b_0 D_4) w_{,xxssss} +$$

$$
\begin{aligned}
&+(a_2 b_5 c_2 + a_5 c_2 + b_3 c_4 + a_2 b_3 c_5 + a_1 b_2 c_9 + a_2 b_1 c_9 - c_9)\, W_{,xxxss} \\
&+(a_5 b_1 c_1 + b_5 c_1 + a_1 b_5 c_3 + a_3 b_1 c_4 + a_1 b_4 c_5 + a_3 c_5 + a_1 b_2 c_7 + a_2 b_1 c_1 - c_7 \\
&+ a_1 b_1 c_8 + a_2 b_0 D_1 + 2a_1 b_0 D_3 + 2a_1 b_0 D_4)\, W_{,xxxxss} + (a_1 b_1 c_2 + a_1 b_5 c_5 \\
&+ a_1 b_1 c_9)\, W_{,xxxxxs} + (a_1 b_1 c_7 + a_1 b_0 D_1)\, W_{,xxxxxx} + (a_2 b_4 c_3 + a_2 b_2 c_6 \\
&+ a_2 b_0 c_8)\, W_{,ssss} + (a_2 b_4 c_2 + a_2 b_3 c_2 + a_2 b_0 c_9)\, W_{,xsss} + (a_2 b_0 c_1 \\
&+ a_3 b_2 c_1 + b_4 c_1 + a_2 b_3 c_2 + a_1 b_4 c_3 + a_3 c_3 + a_3 b_0 c_4 + a_1 b_2 c_6 \\
&+ a_2 b_1 c_6 - c_6 + a_2 b_0 c_7 + a_1 b_0 c_8)\, W_{,xxss} + (b_3 c_1 + a_1 b_4 c_2 + a_3 c_2 \\
&+ a_1 b_3 c_3 + a_1 b_0 c_9)\, W_{,xxxs} + (a_3 b_1 c_1 + a_1 b_3 c_2 + a_1 b_1 c_6 \\
&+ a_1 b_0 c_7)\, W_{,xxxx} + (a_2 b_0 c_6)\, W_{,ss} + (a_3 b_0 c_1 + b_0 a_1 c_6)\, W_{,xx} = 0
\end{aligned}
$$

The substitution of Equations (3) into Equation (20) results in the following differential equation in which the constant coefficients are expressed in terms of the extensional and shearing stiffnesses α_1 , α_2 , α_3 , and α_4 ; and the bending and twist rigidities D_1 , D_2 , D_3 , and D_4 .

$$
\begin{aligned}
&d_1(R^7 W_{,ssssssss}) + d_2(R^7 W_{,xxssssss}) + d_3(R^7 W_{,xxxxssss}) \\
&+ d_4(R^7 W_{,xxxxxxss}) + d_5(R^7 W_{,xxxxxxxx}) + (Kd_6 + pR^2 d_7)(R^5 W_{,ssssss}) \\
&+(N_{xs} R d_8)(R^5 W_{,xsssss}) + (d_9 + Kd_{10} + pR^2 d_{11} - N_{xx} R d_{12})(R^5 W_{,xxssss}) \\
&+(N_{xs} R d_{13})(R^5 W_{,xxxsss}) + (d_{14} + Kd_{15} + pR^2 d_{16} - N_{xx} R d_{17})(R^5 W_{,xxxxss}) \\
&+(N_{xs} R d_{18})(R^5 W_{,xxxxxs}) + (Kd_{19} + pR^2 d_{20} - N_{xx} R d_{21})(R^5 W_{,xxxxxx}) \\
&+(K^2 d_{22} + pR^2 d_{23} + KpR^2 d_{24})(R^3 W_{,ssss}) + (N_{xs} R d_{25} + N_{xs} p R^3 d_{26})(R^3 W_{,xsss}) \\
&+(d_{27} + K^2 d_{28} + pR^2 d_{29} + KpR^2 d_{30} - N_{xx} p R^3 d_{31} + N_{xs}^2 R^2 d_{32})(R^3 W_{,xxss}) \\
&+(N_{xs} R d_{33})(R^3 W_{,xxxs}) + (d_{34} + K^2 d_{35} + KpR^2 d_{36} - N_{xx} p R^3 d_{37} + N_{xs}^2 R^2 d_{38})(R^3 W_{,xxxx}) \\
&+(pR^2 d_{39} + K^2 pR^2 d_{40})(R W_{,ss}) + (pR^2 d_{41} + K^2 p R^2 d_{42})(R W_{,xx}) = 0 \qquad (21)
\end{aligned}
$$

where

$$d_1 = \alpha_2\alpha_3 D_2 R + (\alpha_2 D_2 D_3)/(2R)$$

$$d_2 = R(\alpha_1\alpha_2 D_2 + 2\alpha_2\alpha_3 D_3 + 2\alpha_2\alpha_3 D_4) + (\alpha_2 D_3 D_4)/R$$

$$\begin{aligned} d_3 = {} & 2\alpha_1\alpha_2 D_3 + 2\alpha_1\alpha_3 D_4 R + \alpha_2\alpha_3 D_1 R + \alpha_1\alpha_3 D_2 R - 2\alpha_4^2 D_3 R - 4\alpha_3\alpha_4 D_3 R \\ & - 2\alpha_4^2 D_4 R - 4\alpha_3\alpha_4 D_4 + (\alpha_2 D_1 D_3)/(2R) + (\alpha_1 D_2 D_3)/(2R) + (4\alpha_3 D_3 D_4)/R \\ & + (2\alpha_4 D_3 D_4)/R \end{aligned}$$

$$\begin{aligned} d_4 = {} & \alpha_1\alpha_2 D_1 R + 2\alpha_1\alpha_3 D_3 R + 2\alpha_1\alpha_3 D_4 R - \alpha_4^2 D_1 R - 2\alpha_3\alpha_4 D_1 R + (\alpha_1 D_3 D_4)/R \\ & + (2\alpha_3 D_1 D_3)/R + (\alpha_4 D_1 D_3)/R \end{aligned}$$

$$d_5 = \alpha_1\alpha_3 D_1 R + (\alpha_1 D_1 D_3)/(2R)$$

$$d_6 = 2\alpha_2\alpha_3 D_2 R + (\alpha_2 D_2 D_3)/R$$

$$d_7 = \alpha_2\alpha_3 R^2 + \alpha_3 D_2 + (\alpha_2 D_3)/2 + (D_2 D_3)/(2R^2) \tag{22}$$

$$d_8 = -(2\alpha_3\alpha_2 R^2 + \alpha_2 D_3)$$

$$d_9 = (4\alpha_2\alpha_3 D_3)R + (2\alpha_4 D_2 D_3)/R$$

$$\begin{aligned} d_{10} = {} & 2\alpha_1\alpha_2 D_2 R - 2\alpha_3\alpha_4 D_2 R + 2\alpha_2\alpha_3 D_4 R - (2\alpha_4^2 D_2)R + (2\alpha_3 D_2 D_3)/R \\ & + (\alpha_2 D_3 D_4)/R \end{aligned}$$

$$d_{11} = \alpha_1\alpha_2 R^2 - \alpha_4^2 R^2 - 2\alpha_3\alpha_4 R^2 + \alpha_1 D_2 - \alpha_4 D_3 + 2\alpha_3 D_4 + (D_3 D_4)/R^2$$

$$d_{12} = \alpha_2\alpha_3 R^2 + (\alpha_2 D_3)/2$$

$$d_{13} = 4\alpha_3\alpha_4 R^2 - 2\alpha_1\alpha_2 R^2 + 2\alpha_4^2 R^2$$

$$d_{14} = 2\alpha_1\alpha_2 D_3 R - 4\alpha_3\alpha_4 D_3 R - 2\alpha_4^2 D_3 R$$

$$\begin{aligned} d_{15} = {} & 2\alpha_1\alpha_2 D_4 R + 2\alpha_1\alpha_3 D_2 R - 2\alpha_4^2 D_4 R - 4\alpha_3\alpha_4 D_4 R \\ & + (\alpha_1 D_2 D_3)/R + (4\alpha_3 D_3 D_4)/R + (2\alpha_4 D_3 D_4)/R \end{aligned}$$

$$d_{16} = \alpha_1\alpha_3 R^2 + \alpha_1 D_4 + (\alpha_1 D_3)/2$$

$$d_{17} = \alpha_1\alpha_2 R^2 - \alpha_4^2 R^2 - 2\alpha_3\alpha_4 R^2 + 2\alpha_3 D_3 + \alpha_4 D_3$$

$d_{18} = \alpha_1 D_3 - 2\alpha_1 \alpha_3 R^2 \qquad d_{19} = 2\alpha_1 \alpha_3 D_4 R + (\alpha_1 D_3 D_4)/R \qquad d_{20} = \alpha_1 D_1$

$d_{21} = \alpha_1 \alpha_3 R^2 + (\alpha_1 D_3)/2 \qquad d_{22} = \alpha_2 \alpha_3 D_2 R + (\alpha_2 D_2 D_3)/2R$

$d_{23} = 2\alpha_2 \alpha_3 R^2 + \alpha_2 D_3 \qquad d_{24} = 2\alpha_3 D_2 + (D_2 D_3)/R^2$

$d_{25} = -2\alpha_2 \alpha_3 R^2 - \alpha_2 D_3 \qquad d_{26} = 2\alpha_3 R - \alpha_1 R + D_3/R.$

$d_{27} = 2\alpha_2 \alpha_3 D_3 R$

$d_{28} = \alpha_1 \alpha_2 D_2 R - \alpha_4^2 D_2 R - 2\alpha_3 \alpha_4 D_2 R + (2\alpha_3 D_2 D_3)/R + (\alpha_4 D_2 D_3)/R$

$d_{29} = 2\alpha_1 \alpha_2 R^2 - 2\alpha_3 \alpha_4 R^2 - 2\alpha_4^2 R^2 - \alpha_4 D_3 \qquad d_{30} = 2\alpha_3 D_4 + 2\alpha_1 D_2 + (D_3 D_4)/R^2$

$d_{31} = -\alpha_3 R - D_3/(2R) \qquad d_{32} = \alpha_3 R + (D_3)/(2R)$

$d_{33} = 2\alpha_3 \alpha_4 R^2 + 2\alpha_4^2 R^2 - 2\alpha_1 \alpha_2 R^2 - \alpha_4 D_3$

$d_{34} = \alpha_1 \alpha_2 \alpha_3 R^3 - \alpha_4^2 \alpha_3 R^3 - (\alpha_1 \alpha_2 D_3 R)/2 - (\alpha_4^2 D_3 R)/2$

$d_{35} = \alpha_1 \alpha_3 D_2 R + (\alpha_1 D_2 D_3)/2R \qquad d_{36} = 2\alpha_1 D_4 \qquad d_{37} = \alpha_1 R \qquad d_{38} = -\alpha_1 R$

$d_{39} = \alpha_2 \alpha_3 R^2 + (\alpha_2 D_3)/2 \qquad d_{40} = \alpha_3 D_2 + (D_2 D_3)/(2R^2)$

$d_{41} = \alpha_1 \alpha_2 R^2; \qquad d_{42} = -\alpha_4^2 R^2 + \alpha_1 D_2$

SOLUTION OF THE DONNELL TYPE OF DIFFERENTIAL EQUATION FOR A LONG CYLINDER

The assumption that the radial buckling displacement w has the following form results in a solution of Equation (21).

$$w = A\left[\sin(m\lambda x/R)\cos(nS/R) + \cos(m\lambda x/R)\sin(nS/R)\right] \tag{23}$$

In the above equation A is a constant and the dimensionless parameter, $\lambda = \pi R/L$, is introduced. The above expression does not satisfy the boundary conditions for either a simply supported or a clamped edge shell; therefore, the following derivation should only be used on a long cylindrical shell in which the boundary condition effects are negligible.

The substitution of Equation (23) into Equation (21) is a solution of the stability equation provided a certain relation is satisfied. This relationship results in the following algebraic equation which is the eigenvalue equation of the stability differential equation.

$$
\begin{aligned}
& n^8 d_1 + \lambda^2 m^2 n^6 d_2 + \lambda^4 m^4 n^4 d_3 + \lambda^6 m^6 n^2 d_4 + \lambda^8 m^8 d_5 \\
& - n^6 (K d_6 + p R^2 d_7) - \lambda m n^5 (\bar{N}_{xs} R) d_8 \\
& - \lambda^2 m^2 n^4 (d_9 + K d_{10} + p R^2 d_{11} - \bar{N}_{xx} R d_{12}) - \lambda^3 m^3 n^3 (\bar{N}_{xs} R) d_{13} \\
& - \lambda^4 m^4 n^2 (d_{14} + K d_{15} + p R^2 d_{16} - \bar{N}_{xx} R d_{17}) - \lambda^5 m^5 n (\bar{N}_{xs} R) d_{18} \\
& - \lambda^6 m^6 (K d_{19} + p R^2 d_{20} - \bar{N}_{xx} R d_{21}) + n^4 (K^2 d_{22} + p R^2 d_{23} + K p R^2 d_{24}) \\
& + \lambda m n^3 (\bar{N}_{sx} R)(d_{25} + p R^2 d_{26}) \\
& + \lambda^2 m^2 n^2 (d_{27} + K^2 d_{28} + p R^2 d_{29} + K p R^2 d_{30} - p \bar{N}_{xx} R^3 d_{32} + \bar{N}_{xs}^2 R^2 d_{31}) \\
& + \lambda^3 m^3 n (\bar{N}_{xs} R) d_{33} \\
& + \lambda^4 m^4 (d_{34} + K^2 d_{35} + K p R^2 d_{36} - p \bar{N}_{xx} R^3 d_{37} + \bar{N}_{xs}^2 R^2 d_{38}) \\
& - n^2 (p R^2)(d_{39} + K^2 d_{40}) - \lambda^2 m^2 (p R^2)(d_{41} + K^2 d_{42}) = 0
\end{aligned} \tag{24}
$$

In the preceding equation m and n are integers whose values govern the buckled mode.

The following terms are defined in order to utilize Equation (24) for design when a torque T and an axial compressive force P are applied to an orthotropic cylindrical shell in addition to a pressure p .

$$q = P/\pi R^2 \qquad\qquad q_o = T/\pi R^2 \tag{25}$$

$$K_1 = p/q \qquad\qquad K_2 = q_o/q \tag{26}$$

Since

$$P = -2\pi R \bar{N}_{xx} \qquad \text{and} \qquad T = 2\pi R^2 \bar{N}_{xs} \tag{27}$$

Then

$$\bar{N}_{xx} = -qR/2 \qquad \text{and} \qquad \bar{N}_{xs} = q_o R/2 = K_2 q R/2 \tag{28}$$

The substitution of Equations (26) and (28) into Equation (24) results in the following quadratic expression in terms of the "axial pressure" .

$$R^4 q^2 (K_1 d_{43} + K_1 K_2 d_{44} + K_2^2 d_{45}) + R^2 q (d_{46} + K_1 d_{47} + K_2 d_{48}) + d_{49} = 0 \tag{29}$$

where

$$\begin{aligned}
d_{43} &= (\lambda^2 m^2 n^2 d_{32} + \lambda^4 m^4 d_{37})/2 \\
d_{44} &= (\lambda m n^3 d_{26})/2 \\
d_{45} &= (\lambda^2 m^2 n^2 d_{31} + \lambda^4 m^4 d_{38})/4 \\
d_{46} &= -(\lambda^2 m^2 n^4 d_{12} + \lambda^4 m^4 n^2 d_{17} - \lambda^6 m^6 d_{21})/2 \\
d_{47} &= -(n^6 d_7 + \lambda^2 m^2 n^4 d_{11} + \lambda^4 m^4 n^2 d_{16} + \lambda^6 m^6 d_{20} - n^4 d_{23} \\
&\qquad - \lambda^2 m^2 n^2 d_{29} + n^2 d_{39} + \lambda^2 m^2 d_{41}) \\
&\qquad + K(n^4 d_{24} + \lambda^2 m^2 n^2 d_{30} + \lambda^4 m^4 d_{36}) - K^2 (n^2 d_{40} + \lambda^2 m^2 d_{42}) \\
d_{48} &= -(\lambda m n^5 d_8 + \lambda^3 m^3 n^3 d_{13} + \lambda^5 m^5 n d_{18} + \lambda m n^3 d_{25} + \lambda^3 m^3 n d_{33})/2 \\
d_{49} &= (n^8 d_1 + \lambda^2 m^2 n^6 d_2 + \lambda^4 m^4 n^4 d_3 + \lambda^6 m^6 n^2 d_4 + \lambda^8 m^8 d_5 \\
&\qquad - \lambda^2 m^2 n^4 d_9 - \lambda^4 m^4 n^2 d_{14} + \lambda^2 m^2 n^2 d_{27} + \lambda^4 m^4 d_{34}) \\
&\qquad - K^2 (n^4 d_{22} + \lambda^2 m^2 n^2 d_{28} + \lambda^4 m^4 d_{35}) \\
&\qquad - K (n^6 d_6 + \lambda^2 m^2 n^4 d_{10} + \lambda^4 m^4 n^2 d_{15} + \lambda^6 m^6 d_{19})
\end{aligned} \tag{30}$$

In Equation (29) the buckled mode is described by the integer values of m and n. Since the lowest critical load is desired, the values used in this equation must be chosen so as to minimize q in order to find its smallest positive value that will satisfy Equation (29). This is equivalent to minimizing the energy. The values of m and n for which q will be a minimum in the past have been obtained by the following methods.

1. A value for m was chosen based on experimental evidence and, assuming n continuous, n was mathematically minimized with respect to q. Or a value of n was chosen and m formally minimized.
2. A trial and error procedure was used to determine the values of m and n.

3. Graphical methods were used.

For the above methods to insure accurate results, in most cases the labor involved is tremendous; however, the use of a digital computer reduces this time to only a few minutes. Once a computer program has been written that determines the minimum positive values of q for the parameters K, K_1, and K_2, this program can readily be adapted for use in developing design data. No interaction relationships or equations are needed with such a program.

SOLUTION OF THE DONNELL TYPE OF DIFFERENTIAL EQUATION FOR A SIMPLY SUPPORTED CYLINDRICAL SHELL

The assumption that the radial buckling displacement w has the following form also results in a solution of Equation (21) when $\bar{N}_{xs}$ is zero.

$$w = A \left[\sin (m \lambda x / R) \cos (n s / R) \right] \tag{31}$$

The above expression satisfies the boundary conditions on the w displacement for a simply supported cylindrical shell. The substitution of Equation (31) into Equation (21) is a solution of the stability equation provided again that a certain relationship is satisfied. After considerable mathematical manipulation this relationship results in the following second degree algebraic equation which is the eigenvalue equation of the stability differential equation.

$$R^4 q^2 (K_1 d_{45}) + R^2 q (d_{46} + K_1 d_{47}) + d_{49} = 0 \tag{32}$$

It is easily seen that the above equation can be derived from Equation (29) by setting K_2 equal to zero. Thus any procedure that minimizes q from Equation (29) can also be applied to the case of a simply supported cylindrical shell that is subjected to an axial compressive force P combined with a radial pressure p.

SOLUTION OF THE DONNELL TYPE OF DIFFERENTIAL EQUATION FOR A CYLINDER WITH CLAMPED EDGES

For this particular case N_{xs} is again assumed to be zero and the following substitution is made in Equation (21).

$$W = F\bar{G} \quad \text{where } F = F(x) \text{ and } \bar{G} = \bar{G}(s) \tag{33}$$

Also in the following development superscripts are used to denote the order of the derivative. For example: $F^{IV} = \dfrac{d^4F}{dX^4}$

Using the above substitution and notation Equation (21) can be written as follows.

$$\begin{aligned}
&A_1(\bar{G}^{VIII}/\bar{G}) + A_2(F^{II}/F)(\bar{G}^{VI}/\bar{G}) + A_3(F^{IV}/F)(\bar{G}^{IV}/\bar{G}) \\
&+ A_4(F^{VI}/F)(\bar{G}^{II}/\bar{G}) + A_5(F^{VIII}/F) + A_6(\bar{G}^{VI}/\bar{G}) + A_7(F^{II}/F)(\bar{G}^{IV}/\bar{G}) \\
&+ A_8(F^{IV}/F)(\bar{G}^{II}/\bar{G}) + A_9(F^{VI}/F) + A_{10}(\bar{G}^{IV}/\bar{G}) + A_{11}(F^{II}/F)(\bar{G}^{II}/\bar{G}) \\
&+ A_{12}(F^{IV}/F) + A_{13}(\bar{G}^{II}/\bar{G}) + A_{14}(F^{II}/F) = 0
\end{aligned} \tag{34}$$

where

$A_1 = d_1/R \qquad A_2 = d_2/R$

$A_3 = d_3/R \qquad A_4 = d_4/R$

$A_5 = d_5/R \qquad A_6 = (Kd_6 + K_1 g R^2 d_7)/R^2$

$A_7 = (d_9 + Kd_{10} + K_1 g R^2 d_{11} + gR^2 d_{12}/2)/R^3$

$A_8 = (d_{14} + Kd_{15} + K_1 g R^2 d_{16} + gR^2 d_{17}/2)/R^3$

$A_9 = (Kd_{19} + K_1 g R^2 d_{20} + gR^2 d_{21}/2)/R^3$

$A_{10} = (K^2 d_{22} + K_1 g R^2 d_{23} + KK_1 g R^2 d_{24})/R^5$

$A_{11} = (d_{27} + K^2 d_{28} + K_1 g R^2 d_{29} + KK_1 g R^2 d_{30} + gR^4 d_{31}/2)/R^5$

$A_{12} = (d_{34} + K^2 d_{35} + KK_1 g R^2 d_{36} + gR^4 d_{37}/2)/R^5$

$A_{13} = (d_{39} + K^2 d_{40})(K_1 g R^2/R^7)$

$A_{14} = (d_{41} + K^2 d_{42})(K_1 g R^2/R^7)$

Equation (34) can be written as follows:

$$F_1 + G_1 + F^{II} G_2/F + F^{IV} G_3/F + F^{VI} G_4/F = 0 \tag{36}$$

where

$$F_1 = A_{14} F^{II}/F + A_{12} F^{IV}/F + A_4 F^{VI}/F + A_5 F^{VIII}/F \tag{37}$$

$$G_1 = A_1 \bar{G}^{VIII}/\bar{G} + A_6 \bar{G}^{VI}/\bar{G} + A_{10} \bar{G}^{IV}/\bar{G} + A_{13} \bar{G}^{II}/\bar{G} \tag{38}$$

$$G_2 = A_2 \bar{G}^{IV}/\bar{G} + A_7 \bar{G}^{IV}/\bar{G} + A_{11} \bar{G}^{II}/\bar{G} \tag{39}$$

$$G_3 = A_3 \bar{G}^{IV}/\bar{G} + A_8 \bar{G}^{II}/\bar{G} \tag{40}$$

$$G_4 = A_4 \bar{G}^{II}/\bar{G} \tag{41}$$

After repeated differentiation and considerable algebraic manipulation it can be shown that

$$G_1 = B_1 \tag{42}$$

$$G_2 = B_2 \tag{43}$$

$$G_3 = B_3 \tag{44}$$

$$G_4 = B_4 \tag{45}$$

where B_1, B_2, B_3, and B_4, are constants.

Equations (38), (39), (40), and (41) can also be written in the following manner using Equations (42), (43), (44), and (45).

$$\bar{G}^{II}/\bar{G} = B_4/A_4 \tag{46}$$

$$\bar{G}^{IV}/\bar{G} = B_3/A_3 - (A_8 B_4)/(A_3 A_4) \tag{47}$$

$$\bar{G}^{VI}/\bar{G} = B_2/A_2 + (A_7 A_8 B_4)/(A_2 A_3 A_4) - (A_3 A_7 B_3)/A_2 - (A_{11} B_4)/(A_2 A_4) \tag{48}$$

$$\bar{G}^{VIII}/\bar{G} = B_1/A_1 + (A_3 A_6 A_7 B_3)/(A_1 A_2) + (A_6 A_{11} B_4)/(A_1 A_2 A_3) - (A_6 B_2)/(A_1 A_2) - (A_6 A_7 A_8 B_4)/(A_1 A_2 A_3 A_4) \tag{49}$$

Since $\bar{G}$ is a function of the variable that represents the circumferential direction, it must be a periodic function. Therefore, the solution of Equations (46), (47), (48), and (49) is given by trigonometric functions. As a result the following values for the constants B_1, B_2, B_3, and B_4, are obtained.

$$B_1 = n^8 A_1/R^8 - n^6 A_6/R^6 + n^4 A_{10}/R^4 - n^2 A_{13}/R^2 \tag{50}$$

$$B_2 = -n^6 A_2/R^6 + n^4 A_7/R^4 - n^2 A_{11}/R^2 \tag{51}$$

$$B_3 = n^4 A_3/R^4 - n^2 A_8/R^2 \tag{52}$$

$$B_4 = n^2 A_4/R^2 \tag{53}$$

The substitution of Equations (37), (42), (43), (44), (45), (50), (51), (52), and (53), into Equation (36) results in the following differential equation.

$$F^{VIII} + B_5 F^{VI} + B_6 F^{IV} + B_7 F^{II} + B_8 F = 0 \tag{54}$$

where

$$B_5 = (A_9 R^2 - n^2 A_4)/(A_5 R^2) \tag{55}$$

$$B_6 = (A_{12} R^4 - n^2 A_8 R^2 + n^4 A_3)/(A_5 R^4) \tag{56}$$

$$B_7 = (A_{14} R^6 - n^2 A_{11} R^4 + n^4 A_7 R^2 - A_2 n^6)/(A_5 R^6) \tag{57}$$

$$B_8 = (-n^2 A_{13} R^6 + n^4 A_{10} R^4 - n^6 A_6 R^2 + n^6 A_1)\ (A_5 R^8) \tag{58}$$

$$(r^2)^4 + B_5 (r^2)^3 + B_6 (r^2)^2 + B_7 (r^2) + B_8 = 0 \tag{59}$$

When the quantities K_i and q and positive integer values of n are chosen in such a manner that two roots of the preceding quartic equation are real and negative, then a solution of Equation (54) can be written in the following manner.

$$F = B_9 \sin r_1 x + B_{10} \cos r_1 x + B_{11} \sin r_2 x + B_{12} \cos r_2 x \tag{60}$$

where B_9, B_{10}, B_{11}, and B_{12}, are constants and four roots of Equation (59) are written in the form $r = \pm i r_1$ and $r = \pm i r_2$.

If Equation (60) is used to satisfy the following clamped edge geometric boundary conditions, then the natural boundary conditions given by Equations (16) may not be satisfied.

$$F(0) = F(L) = F^{I}(0) = F^{I}(L) = 0 \tag{61}$$

The application of the preceding boundary conditions to Equation (60) results in the following linear homogeneous algebraic equations in terms of the constants given in Equation (60).

$$B_{10} + B_{12} = 0$$

$$B_9 r_1 + B_{11} r_2 = 0 \qquad (62)$$

$$B_9 \sin r_1 L + B_{10} \cos r_1 L + B_{11} \sin r_2 L + B_{12} \cos r_2 L = 0$$

$$B_9 r_1 \cos r_1 L - B_{10} r_1 \sin r_1 L + B_{11} r_2 \cos r_2 L - B_{12} r_2 \sin r_2 L = 0$$

For a non-trivial solution of Equations (62) to exist, the determinant of the coefficients of B_9 , B_{10} , B_{11} , and B_{12} , must be zero or

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ r_1 & 0 & r_2 & 0 \\ \sin r_1 L & \cos r_1 L & \sin r_2 L & \cos r_2 L \\ r_1 \cos r_1 L & -r_1 \sin r_1 L & r_2 \cos r_2 L & -r_2 \sin r_2 L \end{vmatrix} = 0 \qquad (63)$$

From the preceding expression the following eigenvalue equation of the stability differential equation is obtained.

$$(r_1^2 + r_2^2)(\sin r_1 L \sin r_2 L) + 2 r_1 r_2 (\cos r_1 L \cos r_2 L - 1) = 0 \qquad (64)$$

In order to determine the buckling load for a clamped edge circular cylindrical shell subjected to a combination of a hydrostatic pressure and an axial compressive load, by use of the preceding analysis, a digital computer program needs to the written that established sets of values for K_1 and q and positive integer values of n which satisfy Equations (59) and (64) and yield two negative values for r^2 in Equation (59). The minimum positive value of q that satisfies these conditions determines the critical buckling load.

Calculations need to be performed for comparison with experimental results in order to establish the validity of the preceding analysis. If the preceding analysis does not yield accurate comparison with experiments, then a general solution of Equation (54) needs to be found that satisfies the geometric boundary conditions given by Equations (61) and the natural boundary conditions given by Equations (16).

NOTATION

$a_1, b_1, c_1, d_1, A_1, B_1$, etc	Constants that are functions of the extensional and shear stiffnesses, the bending and twist rigidities, the applied pressure, the axial load, and the applied torque
A_s	Area of the middle surface of the shell
D_1, D_2, D_3, D_4	Bending and twist rigidities of an elemental area of an orthotropic circular cylindrical shell
e_{xx}, e_{ss}, e_{xs}	Axial, circumferential and shearing strain
E_x, E_s	Moduli of elasticity for orthotropic circular cylindrical shell
F	Function of the axial coordinate derived from the radial displacement
G	Shear modulus for orthotropic circular cylindrical shell
$\bar{G}$	Function of the circumferential coordinate derived from the radial displacement
h	Wall thickness of the shell
K	Parameter introduced for the purpose of later studying the effect of previously neglected higher order energy terms. For the present paper .
K_1	Ratio of the radial pressure to an axial load function
K_2	Ratio of a torque function to an axial load function
L	Length of the shell
m	Integer that indicates buckled mode in the axial direction
n	Integer that indicates buckled mode in the circumferential direction
$\bar{N}_{xx}, \bar{N}_{ss}, \bar{N}_{xs}$	Axial, circumferential and shear stress resultants per unit length
p	Radial or hydrostatic pressure
$\underline{P}$	Axial load

q	Function of the axial load expressed in pressure units
q_0	Function of the applied torque expressed in pressure units
Q	Mathematical operator
r, r_1, r_2	Roots of an auxiliary equation
R	Radius of cylindrical shell
x, s, z	Axial, circumferential and radial coordinates of cylindrical shell middle surface
T	Applied torque
u, v, w	Axial, circumferential and radial displacements of cylindrical shell middle surface
U	Change in the strain energy of the shell during the buckling process
V	Change in the potential energy of the external forces during the buckling process
V_s	Volume of shell wall
$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	Extensional and shearing stiffnesses of orthotropic cylindrical shell
λ	Parameter that defines the ratio of the radius of the cylinder to the length of the cylinder
ν_{xs}, ν_{sx}	Poisson's ratios for orthotropic shell
$\bar{\sigma}_{xx}, \sigma_{xx}, \bar{\sigma}_{ss}, \sigma_{ss}, \bar{\sigma}_{xs}, \sigma_{xs}$	Axial, circumferential and shearing stresses

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